Finite Math - Spring 2017
Lecture Notes - 4/10/2017

Homework

- Section 4.6 - 9, 12, 14, 15, 17, 18, 21, 26, 29, 32, 37, 39, 41, 42, 45, 46, 55

Section 4.6 - Matrix Equations and Systems of Linear Equations

Matrix Equations.

Theorem 1. *Assume that all products and sums are defined for the indicated matrices A, B, C, I, and 0 (where 0 stands for the zero matrix). Then*

- *Addition Properties*
  (1) *Associative*
  $$(A+B)+C = A+(B+C)$$
  (2) *Commutative*
  $$A+B = B+A$$
  (3) *Additive Identity*
  $$A+0 = 0+A = A$$
  (4) *Additive Inverse*
  $$A+(-A) = (-A)+A = 0$$
- *Multiplication Properties*
  (1) *Associative Property*
  $$A(BC) = (AB)C$$
  (2) *Multiplicative Identity*
  $$AI = IA = A$$
  (3) *Multiplicative Inverse*
  If A is a square matrix and A^{-1} exists, then $AA^{-1} = A^{-1}A = I$
- *Combined Properties*
  (1) *Left Distributive*
  $$A(B+C) = AB+AC$$

(2) *Right Distributive*

$$(B + C)A = BA + CA$$

- *Equality*

  (1) *Addition*
  If $A = B$*, then* $A + C = B + C$

  (2) *Left Multiplication*
  If $A = B$*, then* $CA = CB$

  (3) *Right Multiplication*
  If $A = B$*, then* $AC = BC$

We can use the rules above to solve various matrix equations. In the next 3 examples, we will assume all necessary inverses exists.

Example 1. *Suppose* A *is an* $n \times n$ *matrix and* B *and* X *are* $n \times 1$ *column matrices. Solve the matrix equation for* X

$$AX = B.$$

Solution. *If we multiply both sides of this equation ON THE LEFT by* A^{-1} *we find*

$$A^{-1}(AX) = A^{-1}B \quad \Longrightarrow \quad (A^{-1}A)X = IX = X = A^{-1}B$$

Example 2. *Suppose* A *is an* $n \times n$ *matrix and* B*,* C*, and* X *are* $n \times 1$ *matrices. Solve the matrix equation for* X

$$AX + C = B.$$

Solution. *Begin by subtracting* C *to the other side*

$$AX + C = B \quad \Longrightarrow \quad AX = B - C$$

and now multiply on the left by A^{-1}

$$A^{-1}(AX) = A^{-1}(B - C) \quad \Longrightarrow \quad (A^{-1}A)X = IX = X = A^{-1}(B - C) = A^{-1}B - A^{-1}C$$

Example 3. *Suppose* A *and* B *are* $n \times n$ *matrices and* C *is an* $n \times 1$ *matrix. Solve the matrix equation for* X

$$AX - BX = C.$$

What size matrix is X*?*

Matrix Equations and Systems of Linear Equations. We can also solve systems of equations using the above ideas. These apply in the case that the system has the same number of variables as equations and the coefficient matrix of the system is invertible. If that is the case, for the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & a_{1n}x_n & = & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

We can create the matrix equation

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{1n} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then, if A is invertible (as is the case when the system is consistent and independent, i.e., exactly one solution), we have

$$X = A^{-1}B.$$

Example 4. *Solve the system of equations using matrix methods*

$$\begin{array}{ccccc} x & + & 2y & = & k_1 \\ x & + & 3y & = & k_2 \end{array}$$

where

(a) $k_1 = 1,\ k_2 = 3$

(b) $k_1 = 3,\ k_2 = 5$

(c) $k_1 = -2,\ k_2 = 1$

Solution. *Begin by writing this system as a matrix equation*

$$AX = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = B$$

Our goal is to find $A^{-1}B$, so first find A^{-1}:

$$A^{-1} = \frac{1}{3-2} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 3k_1 - 2k_2 \\ -k_1 + k_2 \end{bmatrix}.$$

So, to get the solutions for the different parts, just plug in the given k_1 and k_2's:

(a) $x = -3$ *and* $y = 2$

(b) $x = -1$ *and* $y = 2$

(c) $x = -8$ *and* $y = 3$

Example 5. *Solve the system of equations using matrix methods*

$$\begin{array}{rcrcl} 2x & + & y & = & k_1 \\ 5x & + & 3y & = & k_2 \end{array}$$

where

(a) $k_1 = 2,\ k_2 = 13$

(b) $k_1 = 2,\ k_2 = 4$

(c) $k_1 = 1,\ k_2 = -3$

Solution.

(a) $x = -7$ *and* $y = 16$

(b) $x = 2$ *and* $y = -2$

(c) $x = 6$ *and* $y = -11$